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On a result of Jachymski, Matkowski, and Świątkowski

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Abstract. We prove on a new manner, via Monotone Principle (of fixed point), a result of Jachymski, Matkowski, and Świątkowski [Journal of Applied Analysis, **1** (1995), 125–134, Theorem 1, p. 130].

1. Introduction

In recent years a great number of papers have presented generalizations of the well-known Banach-Picard contraction principle.

Recently, Jachymski-Matkowski-Świątkowski have proved the following statement (see [1, Theorem 1, p. 130]).

Theorem 1 (Jachymski-Matkowski-Świątkowski [1])**.** *Let (X,d) be a Hausdorff semimetric and d-Cauchy complete space such that the diameters of open balls with radius r are equibounded. Let T be a selfmap on X such that*

$$\text{(Jm)} \qquad d(Tx,Ty) \leq \psi\Big(d(x,y)\Big) \quad \textit{for all} \quad x,y \in X$$

where $\psi : \mathbf{R}_+^0 \to \mathbf{R}_+^0 := [0,+\infty)$ is a nondereasing function such that satisfies $\lim_{n\to\infty} \psi^n(t) = 0$ for every $t > 0$. Then T has a unique fixed point $\xi \in X$ and $T^n(z) \to \xi$ $(n \to \infty)$ for every $z \in X$.

We notice that this result is an essential extension of a former result in 1975 of Matkowski [2].

In connection with the preceding, a topological space (X,τ) is *semimetrizable* iff there is a distance function d such that for any $A \subset X$ is $\overline{A} = \{x \in X : d(x,A) = 0\}$. In this case d is said to be a *semimetric.*

Further, a symmetric or semimetric space (X,d) is *d-Cauchy complete* if every d-Cauchy sequence is *τ-convergent.* Otherwise, a sequence $\{x_n\}_{n\in\mathbf{N}}$ is *d-Cauchy* if given $\varepsilon > 0$ there is a $k \in \mathbf{N}$ such that $d(x_n,x_m) < \varepsilon$ for all $n,m \geq k$.

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Also, a *distance function* for a set X is a function d from $X \times X$ into $\mathbf{R}_+^0$ such that $d(x,y)=0$ if and only if $x=y$, and $d(x,y)=d(y,x)$ for all $x,y \in X$. A distance function is also called a *symmetric*.

2. Further facts

Let X be a topological space and $T: X \to X$ a mapping. For $x \in X$, $O(x) := \{x, Tx, T^2x, \ldots\}$ is called the *orbit* of x. A function g mapping X into the reals is *T-orbitally lower semicontinuous* at p if $\{x_n\}_{n\in\mathbf{N}}$ is a sequence in $O(x)$ and $x_n \to p$ $(n \to \infty)$ implies that $g(p) \le \liminf_{n\to\infty} g(x_n)$. A mapping $T: X \to X$ is said to be *orbitally continuous* if $\xi, x \in X$ are such that ξ is a cluster point of $O(x)$, then $T\xi$ is a cluster point of $T(O(x))$.

In connection with the preceding facts we notice that in 1985 and 1990 Tasković proved so-called Monotone Principle (of fixed point) on arbitrary topological spaces.

In this sense, let X be a topological space, $T: X \to X$, and let $A: X \times X \to \mathbf{R}_+^0$. In 1985 we introduced the concept of TCS-convergence in a space X,i.e., a topological space X satisfies the condition of *TCS-convergence* iff $x \in X$ and if $A(T^n x, T^{n+1}x) \to 0$ $(n \to \infty)$ implies that $\{T^n x\}_{n\in\mathbf{N}}$ has a convergent subsequence. In 1985 and 1990 we have the following result.

Theorem 2 (Monotone Principle, Tasković [4]). *Let T be a mapping of a topological space X into itself, where X satisfies the condition of TCS-convergence. Suppose that there exists a mapping $\varphi: \mathbf{R}_+^0 \to \mathbf{R}_+^0$ satisfying*

$$(\mathrm{I}\varphi) \qquad (\forall t \in \mathbf{R}_+ := (0,+\infty)) \left(\varphi(t) < t \ \text{ and } \ \limsup_{z\to t+0} \varphi(z) < t\right)$$

such that

$$(\mathrm{MP}) \qquad A(Tx,Ty) \le \varphi\big(A(x,y)\big) \quad \text{for all} \quad x,y \in X,$$

where $A: X \times X \to \mathbf{R}_+^0$, $x \mapsto A(x,Tx)$ is T-orbitally lower semicontinuous or T is orbitally continuous, and $A(a,b)=0$ implies $a=b$. Then T has a unique fixed point $\xi \in X$ and $T^n(z) \to \xi$ $(n \to \infty)$ for each $z \in X$.

Proof of Theorem 1. Let $A(x,y)=d(x,y)$, and $\varphi(t)=\psi(t)$. It is easy to see that A and φ, by (MP) and (Iφ), satisfy all the required hypotheses in Theorem 2.

Since d-Cauchy completeness implies TCS-convergence it follows, from Theorem 2, that T has a unique fixed point $\xi \in X$ and that $\{T^n(z)\}_{n\in\mathbf{N}}$ converges to ξ for each $z \in X$. The proof is complete.

An illustration elementary of Theorem 2 is an extension direct of Banach's contraction principle on topological spaces which we give in the following form.

Proposition 1 (Corollary 2 of [4]). *Let T be a mapping of a topological space X into itself, where X satisfies the condition of TCS-convergence. Suppose* *that there exists an* $\alpha \in [0,1)$ *such that*

$$A(Tx, Ty) \leq \alpha A(x,y) \quad \textit{for all} \quad x, y \in X,$$

where $A : X \times X \to \mathbf{R}_+^0$, $x \mapsto A(x, Tx)$ *is* T*-orbitally lower semicontinuous or* T *is orbitally continuous, and* $A(a,b) = 0$ *implies* $a = b$. *Then* T *has a unique fixed point* $\xi \in X$ *and* $T^n(z) \to \xi$ $(n \to \infty)$ *for each* $z \in X$.

The proof of this statement based on Monotone Principle (of fixed point) may be found in [4]. Also, the proof of this statement we can give and elementary without of Theorem 2 in usual manner.

References

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